Ring Theory 6

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Q) Let X be a non-empty set and $R = 2^X$ (power set of X). Is $(R, \cup, \cap)$ a ring?

Let $(R, \cup, \cap)$ be a ring

Ans:- $(R, \cup)$ should be a group

$\Rightarrow \phi$ is the identity element in $(R, \cup)$

$R \ni A \neq \phi$ then we must have A' such that $A \cup A' = \phi$ but it is not possible.

$\Rightarrow\Leftarrow$ So $(R, \cup, \cap)$ is not a ring

Q) Let $(G, +)$ be an abelian group with product defined in G as $ab = 0$ $\forall a, b \in G$. Show that G is a ring.

Ans:- Simple associativity & closed

Q) Let X be a non empty set. Show that the power set $R = 2^X$ with the set operations of symmetric difference $A + B := A \Delta B = (A \cap \bar{B}) \cup (B \cap \bar{A})$ and intersection $AB = A \cap B$, is a ring. Is R commutative? Does R has a unity?

Ans:- $A + (B + C) = (A \cap \overline{B+C}) \cup ((B+C) \cap \bar{A})$

$= (A \cap ((B \cap \bar{C}) \cup (\bar{B} \cap C))) \cup (\bar{A} \cap ((B \cap \bar{C}) \cup (\bar{B} \cap C)))$

$= (A \cap \bar{B} \cap B) \cup (A \cap \bar{B} \cap \bar{C}) \cup (A \cap C \cap B) \cup (A \cap C \cap \bar{C})$
$\cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C)$

$= (A + B) + C$ — associativity holds

Let $A \in R$,

$A + \phi = \phi + A = A$, $\phi \in R$ is the additive identity

$A + A = (A \cap \bar{A}) \cup (\bar{A} \cap \bar{\bar{A}}) = \phi \cup \phi = \phi$ — units holds

$A+A=(A\cap\bar{A})\cup(\bar{A}\cap\bar{\bar{A}}) = \phi\cup\phi=\phi$ — units holds

$A+B=(A\cap\bar{B})\cup(\bar{A}\cap B) = (B\cap\bar{A})\cup(\bar{B}\cap A) = B+A$ — commutativity holds

$A\cdot(BC) = A\cap(B\cap C) = (A\cap B)\cap C$ — associative

$$A(B+C) = A\cap(B+C) = A\cap((B\cap\bar{C})\cup(\bar{B}\cap C))$$
$$= (A\cap B\cap\bar{C})\cup(A\cap\bar{B}\cap C)$$
$$= AB+AC$$

$A\cap X=A \rightarrow X$ is the multiplicative identity $\rightarrow X$ is the unity

So, $(R,\Delta,\cap)$ is a ring.

$AB=A\cap B=B\cap A=BA \longrightarrow R$ is commutative

Q) ϕ is a map from R to S is a ring isomorphism. Show that $Z(R)$ (center of R) is isomorphic to center of S, $Z(S)$. Also, units of R are isomorphic to units of S. under addition.

Ans:- $\phi\cdot R\rightarrow S$

$$Z(R)=\{x : xr=rx \ \forall\ r\in R\}$$

$$Z(S) = \{y : ys=sy \ \forall\ s\in S\}$$

$\forall r\in R$, $\phi(xr) = \phi(x)\phi(r) = \phi(rx) = \phi(r)\phi(x)$

$$\Rightarrow \phi(x)\phi(r) = \phi(r)\phi(x)$$

$$\Rightarrow \phi(x)\in Z(S)$$

$\psi: Z(R)\rightarrow Z(S)$ where $y=\phi(x)$

$x \rightarrow y$

has unique $\phi(x)$ as it will be for $\forall r\in R$ and $\forall s\in S$.

$x \rightarrow y$ where

for bijection, x has unique $\phi(x)$ as it will be for $\forall r \in R$ and $\forall s \in S$.

$$U(R) = \{x ; xr = 1 \text{ for some } r \in R\}$$
$$U(S) = \{y : ys = 1 \text{ for some } s \in S\}$$

$x \in U(R)$

$$\phi(1_R) = 1_S$$
$$\phi(xr) = \phi(1_R) = \phi(x)\phi(r) = 1_S$$
$$\phi(x) \in U(S)$$

$$\psi : U(R) \rightarrow U(S)$$
$$x \rightarrow y \quad \text{where } y = \phi(x)$$

similar reasoning for bijection.